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CONTROL OF THE MOTIONS OF A MEMBRANE BY MEANS OF BOUNDARY FORCES†

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The problem of controlling the transverse motions of a rectangular homogeneous membrane is considered. It is assumed that the force-type controls are solely distributed along its boundary. The known initial and required final distributions of the displacements and velocities of points of the membrane are assumed to be arbitrary, sufficiently smooth functions of the Eulerian coordinates. An initial-boundary-value problem is solved and a problem of moments is formulated. An effective approximate solution is proposed for the mean square integral performance functional. A small numerical parameter, which characterizes the ratio of the greatest period of the free vibrations (of the lowest mode) to the duration of the control process, is taken as a measure of closeness. Laws governing the regulation of the boundary force responses are efficiently constructed and error estimates are obtained. Several practical cases of initial and final conditions, which are frequently encountered in applied problems, are considered.

1. FORMULATION OF THE PROBLEM

The controlled transverse motions of a membrane are considered. Control is exercised by means of forces distributed along its boundary. To be specific, it is assumed that the membrane is homogeneous and rectangular in the plan view (Fig. 1). We shall describe displacements which are orthogonal to the undeformed state by the function $z = z(t,x,y)$, where $t \geqslant 0$ is the time and x, y are the Cartesian (Eulerian) coordinates of the points of the membrane which belong to the rectangle Π with a boundary Γ. The equations of motion of the membrane and the boundary conditions are taken in the following standard form [1, 2]

$$\rho\ddot{z} = \sigma\left(z''_{x^2} + z''_{y^2}\right), \quad (x,y) \in \Pi \setminus \Gamma \tag{1.1}$$

$$\Pi = \{x, y\colon\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\},\ \ \rho, \sigma = \text{const} > 0$$

$$\mp\sigma z'_x|_{x=0,a} = F^{0,a}(t,y),\ \ 0 \leqslant y \leqslant b,\ \ 0 \leqslant t < \infty$$

$$\mp\sigma z'_y|_{y=0,b} = P^{0,b}(t,x),\ \ 0 \leqslant x \leqslant a \tag{1.2}$$

Differentiation with respect to time is indicated by a dot, while differentiation with respect to the x, y-coordinates is indicated using primes with subscripts. The constants ρ and σ have the meaning of the surface density of material and the tension in the membrane, respectively [1].

The equation of state of the membrane (1.1) describes its transverse motions (in particular, vibrations) in the domain Π when there are no external forces distributed over its surface. It is assumed that the forces $F^{0,a} = F^{0,a}(t,y)$, $P^{0,b} = P^{0,b}(t,x)$ are distributed solely along the piecewise-smooth boundary Γ of the rectangular domain Π which is expressed by the boundary relations (1.2), see Fig. 1. The functions $F^{0,a}$, $P^{0,b}$ are unknown and depend on the options proceeding from the aims of the control.

In order to describe the motion of system (1.1), (1.2) and to determine the control functions $F^{0,a}(t, y)$ and $P^{0,b}(t,x)$, which are sufficiently smooth with respect to t, y and t, x, respectively, it is necessary to specify the initial distributions of the displacements z and of the velocities $\dot{z}$ of the points of the membrane (as the result of measurements, for example)

$$z(0,x,y) = u^0(x,y),\ \ \dot{z}(0,x,y) = \upsilon^0(x,y),\ \ (x,y) \in \Pi \tag{1.3}$$

The functions u^0, υ^0 in (1.3) must also be sufficiently smooth. The smoothness properties are discussed

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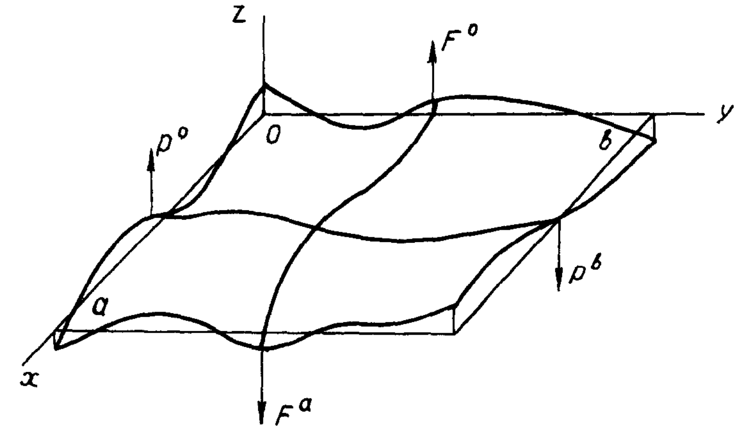

Fig. 1.

more specifically below when constructing the required solution of the control problem and are governed by the conditions for the corresponding Fourier series to converge (see Section 3). For the present, we shall assume that the functions $F^{0,a}$, $P^{0,b}$ and u^0, υ^0 are such that a strong (physical) solution $z(t,x,y)$ of the initial-boundary-value problem (1.1)–(1.3) exists. It is also assumed that the solution converges with respect to an energy norm (the norm W_2^1), that is, that the series for z converges uniformly and that the series for $\dot{z}$, z'_x, z'_y converge with respect to the norm L_2 [1–4].

We will now formulate the purpose of the membrane motion control and the constraints. We will state the problem of controlling the membrane motions by choosing boundary forces $F^{0,a}(t, y)$, $P^{0,b}(t,x)$ from a permissible class of smoothness. It is required that system (1.1)–(1.3) should be brought to a specified state after a finite time

$$z(t_f,x,y) = u^f(x,y),\ \ \dot{z}(t_f,x,y) = \upsilon^f(x,y),\ \ (x,y)\in\Pi,\ \ t_f<\infty \tag{1.4}$$

Here, u^f, υ^f are sufficiently smooth, known functions of x and y. The instant of time t_f in (1.4) when the control process terminates is assumed to be fixed; its value is chosen from certain additional considerations associated with the possibilities of the control system and other factors.

Constraints are usually imposed on the control functions $F^{0,a}$, $P^{0,b}$. These may be, for example, of the geometric type (with respect to a value), with respect to an integral norm (the l-problem of moments [5]) and so on. These constraints may also take the form of integral control performance functionals such as root mean square functionals [5–8], for example. To be specific, we shall take the simplest form of such a constraint [6]

$$I[F,P] = \frac{1}{2}\int_0^{t_f} dt\left[c_F^2\int_0^b F^2(t,y)dy + c_P^2\int_0^a P^2(t,x)dx\right] \to \min_{F,P}$$

$$F = (F^0, F^a)^T,\ \ P = (P^0, P^b)^T,\ \ c_{F,P}^2 = \text{const} > 0 \tag{1.5}$$

Here, F and P are two-dimensional arithmetic vector-functions and c_F^2, c_P^2 are weighting factors. No additional constraints, apart from smoothness conditions, are imposed on the controls $F^{0,a}$, $P^{0,b}$.

The greatly simplified formulation of the problem of controlling the motions of a membrane and their optimization, which has been proposed in the form of (1.1)–(1.5), is of particular theoretical interest and may turn out to be useful in applications, such as in the control of large-scale space structures. The solution of the problem of controllability [5–9] and the construction of the control laws are of considerable importance.

We shall now briefly comment on the boundary (boundary value) conditions (1.2), the initial conditions (1.3) and the final conditions (1.4) as well as on the functional (1.5). According to (1.2), the forces F and P can lead to accelerated translational and rotational motions of the membrane as a whole. In order to avoid large rotations of the plane of the membrane, the functions $F^{0,a}(t,y)$ and $P^{0,b}(t,x)$ must comply with the conditions that there is no total moment of the forces about the axes lying in the plane of the undeformed membrane. In particular, the identities $F_0^0(t) \equiv F_0^a(t), P_0^0(t) \equiv P_0^b(t)$ must be satisfied in the case of the zeroth harmonics (the mean values). Furthermore, in order to avoid significant displacements as a consequence of the control, the boundary forces must be constrained by the requirement that the total value is equal to zero for all t, $0 \leq t \leq t_f$.

We will now discuss the initial conditions (1.3) and the final conditions (1.4). If, at a certain instant of time $t = t^*$, the distributions of the displacements $z(t^*, x, y) \equiv u^* = \text{const}$ and the velocities $\dot{z}(t^*, x, y) \equiv \upsilon^* = \text{const}$ turn out to be homogeneous then, on putting $F = P \equiv 0$ in (1.2) for $t > t^*$, we obtain, according to (1.1), the expressions $z(t, x, y) \equiv u^* + \upsilon^*(t - t^*), \dot{z}(t^*, x, y) \equiv \upsilon^*$. This is a state of uniform motion of the membrane without relative vibrations, which may be of practical interest.

The following formulations of the problem of controlling the transverse motion of a membrane can therefore be of practical importance.

1. Suppression of the relative (elastic) displacements of a membrane in the case of arbitrary initial distributions $u^o(x, y), \upsilon^o(x, y)$ (1.3) from a sufficiently high class of smoothness. The state of translational motion of the membrane as a whole in the direction of the z-axis, that is, the values of u^*, υ^* can be (a) substantial and specified in advance, (b) insubstantial and determined while solving the main problem of the suppression of transverse vibrations.

2. Bringing a membrane from the initial undeformed state (u^o, υ^o are constants) into a state of specified uniform motion $u^f(x, y) = \text{const}$, $\upsilon^f(x, y) = \text{const}$ (see above). This formulation of the control problem can be considered as a special case of the preceding 1(a), when there are no initial relative displacements, or as a supplement to 1(b) (see Section 4).

3. The excitation or (and) suppression of certain selected modes of natural vibration of a membrane with or without taking account of the state of its motion as a whole (see below).

Other particular formulations of the problem of controlling the transverse displacements of a membrane which are of practical interest are also possible within the framework of conditions (1.3) and (1.4). We note that bringing a membrane from an arbitrary initial state $u^o(x, y)$, $\upsilon^o(x, y)$ (1.3) to a required final state $u^f(x, y)$ $\upsilon^f(x, y)$ (1.4) in a restricted time by means of controls concentrated on the boundary in accordance with (1.2) leads to considerable difficulties associated with the controllability and solvability of the finite-dimensional problem of moments [5–9] (see below). It is therefore customary to drop the requirement that the final conditions (1.4) must be strictly satisfied and to take account of them by means of the "penalty method". Methods based on the analytic construction of controllers [10], a finite mode approach, and other methods are also used. A constructive method for solving problem (1.1)–(1.5) is proposed based on the separation of the modes when the time t_f is asymptotically long compared with the period of the natural vibrations of the lowest mode. Such a situation often arises in practical problems.

2. SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR KNOWN CONTROLS

In problem (1.1)–(1.5) it is more convenient to change to the dimensionless variables t_*, x_*, y_*, z_* and parameters by introducing the unit of time θ and the unit of length d. Putting $\theta = d(\rho/\sigma)^{1/2}$, we obtain an equation of state of the form of (1.1) in which $\rho = \sigma = 1$. Here, the value of d can be put equal to a or b and the corresponding variable x_* or y_* varies within the interval [0, 1]. If one carries out a symmetric normalization on the unit of length $d = \alpha a + \beta b$, $\alpha + \beta > 0$, then $\alpha a_* + \beta b_* = 1$; when $d = (a^2 + b^2)^{1/2}$, we have $a_*^2 + b_*^2 = 1$ and if one puts $d = (ab)^{1/2}$, we have $a_* b_* = 1$ and so on.

We now introduce the dimensionless arguments t_*, x_*, y_*, functions $z_*, F_*, P_*, u_*^{0,f}, \upsilon_*^{0,f}$ and parameters $t_{f*}, a_*, b_*, c_{F*}^2, c_{P*}^2$ and omit the asterisk for brevity. We will now solve the initial-boundary-value problem (1.1)–(1.3) under the assumption that the functions $F^{0,a}(t, y)$, $P^{0,b}(t, x)$ are already known.

We shall firstly consider the corresponding eigenvalue and eigenfunction problem which is obtained by using the standard procedure for the separation of variables and the Fourier method [1, 2]. We have

$$Z''_{x^2} + Z''_{y^2} + \lambda^2 Z = 0, \quad z(t, x, y) \sim T(t) Z(x, y)$$
$$Z'_x(0, y) = Z'_x(a, y) = Z'_y(x, 0) = Z'_y(x, y) = 0 \tag{2.1}$$

For the complete orthonormalized system of eigenfunctions $\{Z_{nm}(x, y)\}$ and the system of eigenvalues $\{\lambda_{nm}\}$, we find the expressions

$$Z_{nm}(x, y) = X_n(x) Y_m(y), \quad \lambda_{nm} = (\nu_n^2 + \mu_m^2)^{1/2}$$

$$X_n(x) = (a/2)^{-1/2}\cos\nu_n x,\ X_0(x) = a^{-1/2},\ \nu_n = \pi n/a,\ n \geqslant 1 \qquad (2.2)$$
$$Y_m(y) = (b/2)^{-1/2}\cos\mu_m y,\ Y_0(y) = b^{-1/2},\ \mu_m = \pi m/b, m \geqslant 1$$

Here $\{X_n\}$, $\{\nu_n\}$ and $\{Y_m\}$, $\{\mu_m\}$ are the analogous (complete, orthonormalized) systems for the one-dimensional problems in the intervals $x \in [0, a]$ and $y \in [0, b]$, respectively.

The solution of the boundary-value problem (1.1), (1.2) is constructed using the Fourier method [1, 2] and Grinberg's method [6, 11]. We represent the required function $z(t, x, y)$ in the form of a double series

$$z(t,x,y) = \sum_{k,l \geqslant 0} T_{kl}(t) Z_{kl}(x,y) \qquad (2.3)$$

in which the functions $Z_{kl}(x, y)$ are constructed according to (2.2) and the Fourier coefficients $T_{kl}(t)$ are unknown and have to be determined. On substituting series (2.3) into the left-hand side of Eq. (1.1), multiplying by $Z_{nm}(x, y)dxdy$ and integrating by parts (over the domain Π), we obtain, when account is taken of the boundary conditions, a two-dimensional denumerable system of equations of the form

$$\ddot{T}_{nm} + \lambda_{nm}^2 T_{nm} = \Theta_{nm}(t),\ \Theta_{nm}(t) = (\xi_n, F_m(t)) + (\eta_m, P_n(t))$$
$$\xi_n = (\xi_n^0, \xi_n^a)^T,\ \xi_n^0 = X_n(0),\ \xi_n^a = X_n(a),\ n \geqslant 0 \qquad (2.4)$$
$$\eta_m = (\eta_m^0, \eta_m^b)^T,\ \eta_m^0 = Y_m(0),\ \eta_m^b = Y_m(b),\ m \geqslant 0$$

In Eq. (2.4) for T_{nm}, the expressions of the type (ξ_n, F_m) are scalar products of the corresponding two-dimensional arithmetic vectors which are defined below and in (1.5) and (2.4). The functions $F_m(t)$, $P_n(t)$ are the Fourier coefficients of the boundary controls, that is

$$F(t,y) = \sum_{m \geqslant 0} F_m(t) Y_m(y),\ F_m(t) = \langle F, Y_m \rangle \qquad (2.5)$$

$$P(t,x) = \sum_{n \geqslant 0} P_n(t) X_n(x),\ P_n(t) = \langle P, X_n \rangle$$

Henceforth, the corresponding scalar products in L_2 are denoted by angle brackets.

Equations (2.4) must be supplemented with the initial values of the functions T_{nm}, $\dot{T}_{nm}$. Using the initial conditions (1.3), we obtain the expressions

$$T_{nm}^0 = U_{nm}^0 = \langle u^0, Z_{nm} \rangle,\ \dot{T}_{nm}^0 = V_{nm}^0 = \langle \upsilon^0, Z_{nm} \rangle \qquad (2.6)$$

for the required $T_{nm}(0) = T_{nm}^0$, $\dot{T}_{nm}(0) = \dot{T}_{nm}^0$.

Here U_{nm}^0, V_{nm}^0 are the Fourier components of the known, sufficiently smooth functions with respect to the orthonormalized system $\{Z_{nm}\}$. A strong or classical solution of the initial-boundary-value problem (1.1)–(1.3) can therefore be constructed for known functions $F(t, y)$, $P(t, x)$ of a sufficiently high order of smoothness. It has the form of the double series (2.3) in which the functions T_{nm}, $\dot{T}_{nm}$ are found by the quadratures

$$T_{nm}(t) = U_{nm}^0 \cos\lambda_{nm} t + \frac{V_{nm}^0}{\lambda_{nm}} \sin\lambda_{nm} t + \frac{1}{\lambda_{nm}} \int_0^t \sin\lambda_{nm}(t-\tau)\Theta_{nm}(\tau)d\tau,\ n+m \geqslant 1$$
$$T_{00}(t) = U_{00}^0 + V_{00}^0 t + \int_0^t (t-\tau)\Theta_{00}(\tau)d\tau,\ \dot{T}_{nm} = \frac{dT_{nm}}{dt} \qquad (2.7)$$

The coefficients $T_{nm}(t)$, $n + m \geqslant 1$ characterize the relative displacements of the elements of the membrane (the transverse vibrations). The function $T_{00}(t)$ describes the motion of the membrane as a whole along the z-axis. If the total force $\Theta_{00}(t) \equiv 0$, then the centre of mass of the membrane moves uniformly with an initial velocity $V_{00}^0 Z_{00}$. However, the functions $\Theta_{nm}(t)$ in (2.7) are unknown and are to be determined on the basis of the final conditions (1.4), taking account of the control performance criterion (1.5).

We shall now discuss the frequency properties of the finite-dimensional vibrational system (2.4). It can be shown that the two-dimensional spectrum $\{\lambda_{nm}\}$ forms a dense set in the following sense. By choosing sufficiently large integers of $n \gg 1$ for some $m \leqslant M$, it is possible to achieve any closeness of λ_{nm} to an arbitrary number λ_{nm^*}, where the number $m^* \leqslant M$ is fixed and as large a set of values λ_{nm} as may be desired is contained in the neighbourhood of λ_{nm^*}, which is as small as may be desired. The analogous assertion holds in the case of sufficiently large m for n, $n^* \leqslant N$. Moreover, there is a denumerable set of identical values of λ_{nm} and λ_{kl} in the case of a rational ratio a/b. In particular, when $a = b$ (a square membrane), the frequencies will be identical (apart from the inversion $n \to m$, $m \to n$)

1) $n = 5, m = 5$;	$n = 7, m = 1$;	2) 7, 4; 8, 1;
3) 7, 6; 9, 2;	4) 9, 7; 11, 3;	5) 9, 8; 12, 1;
6) 10, 5; 11, 2;	7) 10, 10; 14, 2;	8) 11, 7; 13, 1;
9) 11, 8; 13, 4;	10) 11, 10; 14, 5;	11) 12, 11; 16, 3;
12) 13, 6; 14, 3;	13) 13, 9; 15, 5;	14) 13, 11; 17, 1;
15) 13, 13; 17, 7;	16) 14, 8; 16, 2;	17) 14, 12; 18, 4;
18) 14, 13; 19, 2;	19) 15, 10; 17, 6; 18, 1;	20) 16, 7; 17, 4;
21) 16, 11; 19, 4;	22) 16, 13; 19, 8; 20, 5;	23) 16, 15; 20, 9;
24) 17, 6; 18, 1;	25) 17, 9; 19, 3;	26) 17, 11; 19, 7;
27) 19, 8; 20, 5;	... and so on	

When $a \gg b$ or $b \gg a$, the frequencies $\lambda_{nm}(a, b)$ are also close for $n, m \sim 1$. The above-mentioned behaviour of the frequency spectrum $\{\lambda_{nm}(a, b)\}$ of system (2.4) makes the effective solution of the control problem [5–9, 12] and the construction of the control laws [5, 6, 10, 13] far more difficult. These difficulties result from the fact that there are no controls distributed over the surface of the membrane, that is, the coefficients $\Theta_{nm}(t)$ in (2.4) do not form a "complete basis".

3. APPROXIMATE SOLUTION OF THE OPTIMAL CONTROL PROBLEM

We will reformulate the initial problem (1.1)–(1.5) in terms of the Fourier coefficients $T_{nm}(t)$ of the function $z(t, x, y)$. The transverse motions of the membrane are described by the system of equations (2.4), (2.5) in which $F_m^{0,a}(t)$, $P_n^{0,b}(t)$ are unknown control functions which are to be determined; the initial conditions have the form of (2.6). The controls must be selected from a permissible class of smoothness such that the final conditions are satisfied at the instant of time $t = t_f$ (1.4)

$$T_{nm}(t_f) = U^f_{nm} = \left\langle u^f, Z_{nm}\right\rangle, \quad \dot{T}_{nm}(t_f) = V^f_{nm} = \left\langle \upsilon^f, Z_{nm}\right\rangle \tag{3.1}$$

The Fourier coefficients $T_{nm}(t)$, $V_{nm}(t) \equiv \dot{T}_{nm}(t)$ are calculated according to (2.7). Moreover, the functions $T_m(t)$, $P_n(t)$ must be bounded in accordance with the optimality condition (1.5) which takes the form (see (2.5))

$$I[F, P] = \frac{1}{2}\int_0^{t_f}\left[c_F^2 \sum_{l\geqslant 0} F_l^2(t) + c_P^2 \sum_{k\geqslant 0} P_k^2(t)\right]dt \to \min_{\{F_m\},\{P_n\}} \tag{3.2}$$

We now apply an analogue of the maximum principle conditions [6, 7] to the finite-dimensional problem of the optimal control (2.4), (2.6), (3.1) and (3.2). We introduce the finite-dimensional vectors $\Phi_{nm}(t)$, $\Psi_{nm}(t)$ associated with the variables T_{nm}, V_{nm} respectively and, using the standard procedure, obtain the expressions for the optimal controls F_m, P_n

$$F_m(t) = c_F^{-2} \sum_{k\geqslant 0} \xi_k \Psi_{km}(t), \quad P_n(t) = c_P^{-2} \sum_{l\geqslant 0} \eta_l \Psi_{nl}(t) \tag{3.3}$$

$$\Psi_{nm}(t) = A_{nm}\sin\lambda_{nm}t + B_{nm}\cos\lambda_{nm}t, \quad n + m \geqslant 1$$

$$\Psi_{00}(t) = A_{00}t + B_{00}, \quad A_{nm}, B_{nm} = \text{const}$$

The coefficients A_{nm}, B_{nm} must satisfy the denumerable system of linear algebraic equations which follow in an elementary manner from (3.1) after substituting the functions $F_m^{0,a}(t)$, $P_n^{0,b}(t)$ (3.3) in formulae (2.7) taking account of expression (2.4) for $\Theta_{nm}(t)$ ($n, m \geqslant 0$), and take the form

$$\sum_{k\geq 0}\left(\xi^{ss}_{nkm}A_{km}+\xi^{sc}_{nkm}B_{km}\right)+\sum_{l\geq 0}\left(A_{nl}\eta^{ss}_{nlm}+B_{nl}\eta^{sc}_{nlm}\right)=$$
$$=U^f_{nm}-U^0_{nm}\cos\lambda_{nm}t_f-(V^0_{nm}/\lambda_{nm})\sin\lambda_{nm}t_f \tag{3.4}$$
$$\sum_{k\geq 0}\left(\xi^{cs}_{nkm}A_{km}+\xi^{cc}_{nkm}B_{km}\right)+\sum_{l\geq 0}\left(A_{nl}\eta^{cs}_{nlm}+B_{nl}\eta^{cc}_{nlm}\right)=$$
$$=V^f_{nm}+U^0_{nm}\lambda_{nm}\sin\lambda_{nm}t_f-V^f_{nm}\cos\lambda_{nm}t_f,\quad n,m\geq 0$$

Coefficients with triple subscripts of the type of $\xi^{ss}_{nkm},\ldots,\eta^{cc}_{nlm}$ (3.4) are obtained by integrating elementary expressions containing products of trigonometric functions or of a trigonometric function and a linear function. For the extreme coefficients, we have

$$\xi^{ss}_{nkm}=\frac{\xi_{nk}}{c_F^2\lambda_{nm}}\int_0^{t_f}\sin\lambda_{nm}(t_f-t)\sin\lambda_{km}t\,dt,\quad n,k\geq 0 \tag{3.5}$$
$$\eta^{cc}_{nlm}=\frac{\eta_{lm}}{c_P^2}\int_0^{t_f}\cos\lambda_{nm}(t_f-t)\cos\lambda_{nl}t\,dt,\quad m,l\geq 0$$

The intermediate coefficients $\xi^{sc}_{nkm},\ldots,\eta^{cs}_{nlm}$ (3.4) have a similar form. There is no need to write out the exceedingly long explicit expressions for the quadratures (3.5) and others. Equations (3.4) and the coefficients (3.5), when $n=m=0$, are obtained by taking the limit as $\lambda_{00}\to 0$.

It is not possible to construct the exact solution of system (3.4). However, analysis of the coefficients (3.5) shows that their asymptotic behaviour with respect to the value of t_f, $t_f\to\infty$ is different and that the diagonal ("resonance") terms are the leading terms. This enables us to construct an approximate solution and to estimate the error with respect to the small quantity $1/t_f\to 0$ [13]. This approach is equivalent to the use of asymptotic methods of the separation of variables which are analogous to the method of averaging [14].

The relations presented above are extremely complex and inconvenient for analysis. It is preferable to use equations in osculating variables [14, 15] which are obtained by means of the standard substitution

$$T_{nm}=C_{nm}\cos\lambda_{nm}t+S_{nm}\sin\lambda_{nm}t,\quad T_{00}\equiv T_{00} \tag{3.6}$$
$$\dot T_{nm}=V_{nm}=\partial T_{nm}/\partial t,\quad V_{00}\equiv V_{00},\quad n+m\geq 1$$

Next, by virtue of Eqs (2.4), on differentiating expressions (3.6) we obtain equations of motion in the variables T_{00}, V_{00}, C_{nm}, S_{nm}, $n+m\geq 1$. The initial conditions (2.6) and final conditions (3.1) are reduced to the required form and, as a result, we have the finite-dimensional two-point problem

$$\dot T_{00}=V_{00},\ \dot V_{00}=\Theta_{00},\quad T_{00}(0)=U^0_{00},\quad V_{00}(0)=V^0_{00}$$
$$\dot C_{nm}=-\lambda^{-1}_{nm}\Theta_{nm}\sin\lambda_{nm}t,\quad \dot S_{nm}=\lambda^{-1}_{nm}\Theta_{nm}\cos\lambda_{nm}t,\quad C_{nm}(0)=U^0_{nm},\quad S_{nm}(0)=\lambda^{-1}_{nm}V^0_{nm} \tag{3.7}$$
$$T_{00}(t_f)=U^f_{00},\quad V_{00}(t_f)=V^f_{00},\quad C_{nm}(t_f)=C^f_{nm}\equiv U^f_{nm}\cos\lambda_{nm}t_f-$$
$$-\lambda^{-1}_{nm}V^f_{nm}\sin\lambda_{nm}t_f,\quad S_{nm}(t_f)=S^f_{nm}\equiv U^f_{nm}\sin\lambda_{nm}t_f+\lambda^{-1}_{nm}V^f_{nm}\cos\lambda_{nm}t_f,\quad n+m\geq 1$$

It follows from (3.7) that, when $\Theta_{nm}(t)\equiv 0$, the variables C_{nm}, $S_{nm}=\text{const}$. The control problem involves the choice of permissible controls F_m, P_n which satisfy the final conditions (3.7) and lead to a minimum of the functional I (3.2). Equations (3.7) are much more convenient since the adjoint variables Φ_{nm}, Ψ_{nm} corresponding to C_{nm}, S_{nm} will be constant (analogous to the coefficients A_{nm}, B_{nm} in $\Phi_{nm}(t)$, $\Psi_{nm}(t)$ (3.3)).

Using the standard maximum principle procedure enables us to obtain the structure of the expressions for the optimal controls as functions of time

$$F_m(t)=\frac{1}{c_F^2}\sum_{k\geq 0}\xi_k\Lambda_{km}(t),\quad P_n(t)=\frac{1}{c_P^2}\sum_{l\geq 0}\eta_l\Lambda_{kl}(t) \tag{3.8}$$
$$\Theta_{nm}(t)=\frac{1}{c_F^2}\sum_{k\geq 0}(\xi_n,\xi_k)\Lambda_{km}(t)+\frac{1}{c_P^2}\sum_{l\geq 0}\Lambda_{nl}(t)(\eta_l,\eta_m)$$

$$\Lambda_{nm}(t)=\lambda_{nm}^{-1}\left(\Psi_{nm}\cos\lambda_{nm}t-\Phi_{nm}\sin\lambda_{nm}t\right)$$

$$\Lambda_{00}(t)=\Psi_{00}^{0}-\Phi_{00}t;\quad \Phi_{00},\Psi_{00}^{0},\quad \Phi_{nm},\Psi_{nm}=\text{const}$$

The coefficients Φ_{nm}, Ψ_{nm}, occurring in the functions Θ_{nm} (3.8), remain to be determined. On substituting these functions into Eqs (3.7) and integrating with respect to t, while taking account of the initial conditions, we obtain the representations for the required variables

$$\begin{aligned}
&V_{00}(t)=V_{00}^{0}+\zeta_{00}^{2}\left(\Psi_{00}^{0}t-\tfrac{1}{2}\Phi_{00}t^{2}\right)+\Delta V_{00}(t),\quad \zeta_{00}^{2}=c_F^{-2}\xi_0^2+c_P^{-2}\eta_0^2\\
&T_{00}(t)=U_{00}^{0}+V_{00}^{0}t+\tfrac{1}{2}\zeta_{00}^{2}\left(\Psi_{00}^{0}t^{2}-\tfrac{1}{3}\Phi_{00}t^{3}\right)+\Delta T_{00}(t)\\
&C_{nm}(t)=U_{nm}^{0}+\tfrac{1}{2}\zeta_{nm}^{2}\lambda_{nm}^{-2}\Phi_{nm}t+\Delta C_{nm}(t),\quad \zeta_{nm}^{2}=c_F^{-2}\xi_n^2+c_P^{-2}\eta_m^2\\
&S_{nm}(t)=\lambda_{nm}^{-1}V_{nm}^{0}+\tfrac{1}{2}\zeta_{nm}^{2}\lambda_{nm}^{-2}\Psi_{nm}t+\Delta S_{nm}(t),\quad n+m\geqslant 1
\end{aligned}\tag{3.9}$$

The leading ("resonance") terms are singled out in expressions (3.9) and the relatively small corrections are denoted by the symbol Δ. We will now determine the unknown coefficients Φ_{nm}, Ψ_{nm} proceeding from the approximate expressions (3.9) in which the corrections are set equal to zero [13]. By solving the equations corresponding to the final conditions (3.7), we find the above-mentioned approximate values of the required coefficients

$$\begin{aligned}
&\Phi_{00}^{(1)}=6\zeta_{00}^{-2}t_f^{-2}(2t_f^{-1}\delta T_{00}-\delta V_{00}),\quad \Psi_{00}^{0(1)}=2\zeta_{00}^{-2}t_f^{-1}(3t_f^{-1}\delta T_{00}-\delta V_{00})\\
&\Phi_{nm}^{(1)}=2\zeta_{nm}^{-2}t_f^{-1}\lambda_{nm}^{2}\delta C_{nm},\quad \Psi_{nm}^{(1)}=2\zeta_{nm}^{-2}t_f^{-1}\lambda_{nm}^{2}\delta S_{nm}\\
&\delta T_{00}=U_{00}^{f}-U_{00}^{0}-V_{00}^{0}t_f,\quad \delta V_{00}=V_{00}^{f}-V_{00}^{0},\quad \delta C_{nm}=C_{nm}^{f}-U_{nm}^{0}\delta S_{nm}=S_{nm}^{f}-\lambda_{nm}^{-1}V_{nm}^{0}
\end{aligned}\tag{3.10}$$

We now introduce a small parameter $\varepsilon=t_f^{-1}\ll 1$; $\varepsilon\to 0$ as $t_f\to\infty$, where t_f is the time of the process measured in units of θ (see the beginning of Section 2). We shall assume the value of Θ_{00} to be small; for example, $\Theta_{00}\sim\varepsilon$ when $0\leqslant t\leqslant t_f$. It suffices to use the estimates $\Phi_{00}^{(1)}\sim\varepsilon^2$, $\Psi_{00}^{0(1)}\sim\varepsilon$ for this purpose, which necessarily hold if $\delta T_{00}\sim\varepsilon^{-1}$, $\delta V_{00}\sim 1$ (see (3.10)). Hence, the sum (resultant) of the boundary forces $F^{0,a}(t,y)$, $P^{0,b}(t,x)$ is assumed to be asymptotically small $O(\varepsilon)$. However, over a long time interval $t_f\sim\varepsilon^{-1}$, the state of motion of the membrane as a whole can vary substantially: $\delta T_{00}\sim\varepsilon^{-1}$ with respect to position and $\delta V_{00}\sim 1$ with respect to velocity. Similarly, we require that the controls Θ_{nm}, $n+m\geqslant 1$ should also be small with respect to the parameter ε, that is, $\Theta_{nm}\sim\varepsilon$. It follows from (3.10) that this requirement is satisfied if $\Phi_{nm}^{(1)}\sim\varepsilon$, $\Psi_{nm}^{(1)}\sim\varepsilon$ or δC_{nm}, $\delta S_{nm}\sim 1$. The mechanical meaning of this assumption is that, by acting ("in resonance") on the edge of the membrane with a force $\Theta_{nm}\sim\varepsilon$ for a long time, it is possible to change (by an amount $O(1)$) the amplitude of the transverse vibrations of an arbitrary mode T_{nm}, $n+m\geqslant 1$, considerably.

Using the approximate values (3.10) of the adjoint variables, we obtain, by (3.8), by controls F_m, P_n, Θ_{nm} in explicit form

$$\begin{aligned}
&F_m^{(1)}(t)=\frac{1}{c_F^2}\sum_{k\geqslant 0}\xi_k\Lambda_{km}^{(1)}(t),\quad P_n^{(1)}(t)=\frac{1}{c_P^2}\sum_{l\geqslant 0}\Lambda_{nl}^{(1)}(t)\eta_l\\
&\Theta_{nm}^{(1)}(t)=\frac{1}{c_F^2}\sum_{k\geqslant 0}(\xi_n,\xi_k)\Lambda_{km}^{(1)}(t)+\frac{1}{c_P^2}\sum_{l\geqslant 0}\Lambda_{nl}^{(1)}(t)(\eta_l,\eta_m)\\
&\Lambda_{00}^{(1)}(t)=\Psi_{00}^{0(1)}-\Phi_{00}^{(1)}t,\quad \Lambda_{nm}^{(1)}(t)=\lambda_{nm}^{-1}\left(\Psi_{nm}^{(1)}\cos\lambda_{nm}t-\Phi_{nm}^{(1)}\sin\lambda_{nm}t\right)
\end{aligned}\tag{3.11}$$

Expressions (3.11) and (3.10) therefore determine the optimal control of the first approximation with respect to the small parameter ε. Indeed, let us substitute the known controls (3.11) into (3.7) and integrate with the specified initial conditions. We obtain explicit representations for the coefficients T_{00}, V_{00}, C_{nm}, S_{nm}, $n+m\geqslant 1$ of the form (3.9) where the constants are defined by (3.10) and the corrections, which are relatively small (with respect to the powers of ε), can be estimated. The possibility of constructing such estimates is closely associated with the convergence of the series, which is determined by the rate of decrease in $\Phi_{nm}^{(1)}$, $\Psi_{nm}^{(1)}$ (3.10), that is, by the classes of smoothness of the functions $u^{0,f}(x,y)$, $v^{0,f}(x,y)$.

It follows from (3.11) that the series for $F^{(1)}_m$, $P^{(1)}_n$, $\Theta^{(1)}_{nm}$ and, also, the series of (2.5) for $F^{(1)}(t, y)$, $P^{(1)}(t,x)$ converges absolutely and uniformly provided that $\Lambda^{(1)}{}_{nm} \sim (nm)^{-(1+\gamma)}$ which, according to (3.10), is equivalent to the conditions δC_{nm}, $\delta S_{nm} \sim (nm)^{-(2+\gamma)}$, where $\gamma > 0$. The conditions are equivalent to the requirements on the Fourier coefficients that $U^{0,f}_{nm}$, $V^{0,f}_{nm}\lambda^{-1}_{nm} \sim (nm)^{-(2+\gamma)}$. We note that estimates of the form $c(nm)^{1/2} \leqslant \lambda_{nm} \leqslant C(n + m)$ hold for λ_{nm}. We shall next assume that the above-mentioned sufficient conditions for convergence are satisfied. The approximate control functions will then be the classical (rather than generalized) functions which can be readily implemented in practice.

We will now estimate the errors in satisfying the final conditions resulting from the quantities Δ in (3.9). It is shown by direct integration of the series for Θ_{00} (without taking account of the leading term) that $\Delta V_{00}(t_f)$, $\sim \varepsilon$, $\Delta T_{00}(t_f) \sim 1$. We note that, with respect to $T_{00}(t_f)$, it is also possible to achieve an error $O(\varepsilon)$. Since $\delta V_{00} \sim 1$, $\delta T_{00} \sim \varepsilon^{-1}$, the relative errors are estimated as $O(\varepsilon)$ [13]. It is then necessary to allow for two facts when estimating the errors $\Delta C_{nm}(t_f)$, $\Delta S_{nm}(t_f)$. The first is the fact that integration of terms of the type $t \sin \lambda_{nm}t$, $t \cos \lambda_{nm}t$ within the limits from $t = 0$ and $t = t_f$ leads to quantities of the order of t_f. Taking account of the factor $O(\varepsilon^2)$, these terms in the above-mentioned series when $n = 0$ or $m = 0$ $(n + m \geqslant 1)$ will be quantities of the order of $O(\varepsilon)$. The second difficulty lies in taking account of the "small denominators" which arise during the integration of terms of the form $\sin \lambda_{km}t \sin \lambda_{nm}t$, $\sin \lambda_{nl}t \sin \lambda_{nm}t$ and similar terms in which there are cosines (see (3.7), (3.8)). The summation is carried out over the indices k and l and, moreover, $k \neq n$, $l \neq m$. The small denominators will be estimated in the following manner

$$|\lambda_{km} - \lambda_{nm}| = (\pi / a)^2 |k - n|(k + n)(\lambda_{km} - \lambda_{nm})^{-1}, \quad |k - n| \geqslant 1$$
$$|\lambda_{nl} - \lambda_{nm}| = (\pi / b)^2 |l - m|(l + m)(\lambda_{nl} + \lambda_{nm})^{-1}, \quad |l - m| \geqslant 1$$

Under the assumptions which have been made regarding the rate of decrease in the coefficients δC_{nm}, δS_{nm}, it can be shown using the integral criterion of the convergence of the series (with respect to k and l) that

$$\Delta C_{nm}(t_f) \sim \varepsilon(nm)^{-2}, \quad \Delta S_{nm}(t_f) \sim \varepsilon(nm)^{-2} \tag{3.12}$$

The occurrence of small denominators does not enable us to increase the order of smallness of the errors with respect to n, m when the rate of decrease in δC_{nm}, δS_{nm} increases, that is, the class of smoothness of the initial and final distributions $u^{0,f}(x,y)$, $\upsilon^{0,f}(x,y)$. It follows from the estimates (3.12) that there is ε-closeness of the solution with respect to a uniform metric and with respect to the L_2 metric of its derivative

$$\max_{(x,\ y)\in\Pi} \left| z^{(1)}(t_f,\ x,\ y) - u^f(x,\ y)\right| \leqslant C\varepsilon, \quad \| \dot z^{(1)}(t_f,\ x,\ y) - \upsilon^f(x,\ y)\|_{L_2} \leqslant C\varepsilon$$

Here, $z^{(1)}(t,x,y)$ is the solution of the boundary-value problem for the known control functions $F^{(1)}(t, y)$, $P^{(1)}(t,x)$ of the first approximation with respect to ε.

A solution of the control problem with a relative error ε with respect to the metric W^1_2 in the class of permissible functions has therefore been constructed. The closeness of the control to the optimal control, that is, with respect to the functional (3.2), requires a special study. It would be expected that the relative error will be an amount $O(\varepsilon^2)$ and the absolute error $O(\varepsilon^3)$ since the adjoint variables are asymptotically small, that is, Φ_{nm}, $\Psi_{nm} = O(\varepsilon)$ and the error in determining them is $O(\varepsilon^2)$. Since the first variation of the functional in the neighbourhood of the optimal value is equal to zero, the correction resulting from the error $O(\varepsilon^2)$ will be a quantity of the second order of magnitude, that is, $\Delta I \sim O(\varepsilon^4)t_f = O(\varepsilon^3)$.

Expressions (3.11) and (3.10) therefore determine the open-loop control in the first approximation with respect to ε. The transition from arbitrary initial distributions $u^0(x, y)$, $\upsilon^0(x, y)$ to the current distributions $z(t, x, y)$, $\dot z(t, x, y)$ at any instant of time t_f is a formal method for constructing a feed-back control. For this purpose, we put $t = 0$ (or, more accurately, $t \to t - t_0$ and then $t_0 \to t$, that is, $t \to 0$) in (3.11); the coefficients $\Phi^{(1)}_{nm}$, $n, m \geqslant 0$ vanish and only $\Psi^{(1)}_{nm}$ remain. We put $t_f \to t_f - t$ (or, more accurately, $t_f \to t_f - t_0$ and then $t_0 \to t$, that is, $t_f \to t_f - t$) for $\Psi^{(1)}_{nm}$ and the Fourier coefficients V^0_{nm} of the initial distribution $\upsilon^0(x, y)$ are replaced by the coefficients V_{nm} of the current velocity distribution $\dot z(t,x,y) = \Sigma V_{kl}(t)Z_{kl}(x,y)$. They can be calculated by processing measurements of $\dot z(t,x,y)$ at the current instant of time $t < t_f$. Note that the controls F_m and P_n possess a singularity of the

$(t_f - t)^{-1}$ type as $t \to t_f$. It is therefore preferable to use other feed-back laws [15] in a small neighbourhood of the terminal manifold $u^f(x, y)$, $v^f(x, y)$.

4. EXAMPLES

We will now consider some special cases of the initial distribution (1.3) and final distribution (1.4) which are frequently encountered in applications. The corresponding formulations of the control problems are presented in Section 1.

1. We will obtain the solution for the suppression of the transverse vibrations of a membrane using forces of the form of (3.11) in which we substitute the quantities $C^f_{nm} = 0$, $S^f_{nm} = 0$, that is, $\delta C_{nm} = -U^0_{nm}$, $\delta S_{nm} = -\lambda^{-1}{}_{nm}V^0_{nm}$ for the parameters $\Phi^{(1)}_{nm}$, $\Psi^{(1)}_{nm}$ (3.10), $n + m \geq 1$. If the conditions of the motion of the membrane as a whole (of the centre of mass) are specified in advance, the parameters $\Phi^{(1)}_{00}$, $\Psi^{0(1)}_{00}$ have the form (3.10) when U^f_{00}, V^f_{00} are fixed. This is case (a). When $t > t_f$, the controls $F^{(1)}(t, y)$, $P^{(1)}(t, x)$ are set equal to zero. In case (b), when the conditions of motion of the centre of mass for $t = t_f$ are unimportant, the controls have the form of (3.11) with the parameters $\Phi^{(1)}_{00} = \Phi^{0(1)}_{00} = 0$, that is, $\Lambda^{(1)}_{00}(t) \equiv 0$. The total or integral values of the forces $F^{(1)}(t, y)$, $P^{(1)}(t, x)$ will be equal to zero. As above, $F^{(1)} = P^{(1)} \equiv 0$ for $t > t_f$ (finite control [5]).

2. Approximate control of the motion of the centre of mass of a membrane $u^0 = \text{const}$, $v^0 = \text{const}$ ($U^0_{nm} = V^0_{nm} = 0$, $n + m \geq 1$) is exercised by means of forces (3.11) in which $\Lambda^{(1)}_{nm}(t) \equiv 0$, $n + m \geq 1$, that is, $F^{(1)}_m(t) = P^{(1)}_n(t) \equiv 0$. The functions $F^{(1)}_0 = c^{-2}_F\xi_0\Lambda^{(1)}_{00}$, $P^{(1)}{}_0 = c^{-2}_P\eta_0\Lambda^{(1)}_{00}$ in which the coefficients $\Phi^{(1)}_{00} = \Psi^{0(1)}_{00}$ are defined by (3.10), are non-zero when $0 \leq t \leq t_f$. This formulation of the problem is a special case of that treated above (see 1a) when $U^0_{nm} = V^0_{nm} = 0$, $n + m \geq 1$. It may also be considered as being supplementary to 1(b) since, after suppression of the vibrations of the membrane, it is brought into a state of specified motion without exciting these vibrations (to the accuracy being considered).

3. Excitation of certain selected modes from the subset of indices $(n, m) \in \{N^* \times M^*\}$ of the natural vibrations of the membrane (or their suppression) with and without allowing for the motion of the centre of mass is performed in the following manner using control forces based on expressions (3.10). The Fourier coefficients U^f_{nm}, V^f_{nm}, $n + m \geq 1$ of the final distributions $u^f(x, y)$, $v^f(x, y)$ for δC_{nm}, δS_{nm} (3.7) must be assumed to be the required values, while the adjoint variables $\Phi^{(1)}_{nm}$, $\Psi^{(1)}_{nm}$, $(n, m) \in \{N^* \times M^*\}$ are calculated from (3.10) and substituted into the functions $\Lambda^{(1)}_{nm}(t)$ (3.11). The remaining modes of vibration $T_{nm}(t)$, $(n, m) \in \{N \times M\} \backslash \{N^* \times M^*\}$ which are of no interest, remain unchanged if one assumes that $\Phi^{(1)}_{nm} = \Psi^{(1)}_{nm} = 0$, that is, $\Lambda^{(1)}_{nm}(t) \equiv 0$ in the case of the above-mentioned indices. If it is required that certain selected modes from the subset of indices $\{N^* \times M^*\}$ should be excited while others from the subset $\{N^{**} \times M^{**}\}$ (all the remaining modes, for example) are suppressed then this problem reduces to the preceding problem $U^f_{nm} = V^f_{nm} = 0$ for the subset $\{N^{**} \times M^{**}\}$). Motion of the centre of mass of the membrane is regulated by the choice of the coefficients $\Phi^{(1)}_{00} = \Psi^{0(1)}_{00}$.

The approximate control laws, constructed in Section 3, therefore enable one to solve in a simple manner the complex problem of the open-loop control of arbitrary modes of the transverse vibrations of a membrane by means of forces distributed along its boundary. Of course, under practical conditions, control of a relatively small number of the lowest vibrational modes can only be performed since generating the high-frequency forces encounters considerable difficulties due to the occurrence of interference.

Note that the problem of controlling the transverse motions of a membrane with arbitrary geometric and physical characteristics (non-rectangular form, inhomogeneous density, non-uniform tension, etc.) can be constructively solved in a similar manner using the approach described in Section 3 [13].

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